Approximating nudging interpolation operator via neural networks

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Data assimilation (DA) plays a critical role in fluid dynamics by effectively integrating observed data with physical models, finding essential applications in weather forecasting, oceanography, and climate modeling. Among DA methods, nudging [1] stands out as a computationally efficient approach and represents a generic framework of a large family of DA in the form of

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x}) - \kappa \left[I(\boldsymbol{x}) - I(\boldsymbol{x}_{obs}) \right], \tag{1}$$

where $\boldsymbol{x}$ is the state variable, whose time derivative is $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$. $F(\cdot)$ is a forward model, which is typically governed by physical equations such as Navier-Stokes equations, and κ is a relaxation parameter, often empirically derived. The observed data, $\boldsymbol{x}_{obs}$ is interpolated using an interpolation operator $I(\cdot)$, which is necessary because observed data is often sparse and incomplete. In addition, interpolation provides rigorous analysis and proof for guaranteed convergence of (1) on the Navier-Stokes equations, and this algorithm is also called Azouani-Olson-Titi (AOT) nudging.

When the relaxation parameter is properly chosen, AOT nudging acts similarly to a simplified Kalman filter, where the Kalman gain degenerates to a constant κ. This significantly improves computational efficiency, albeit at the potential cost of accuracy, while still leaving considerable room for further enhancement. Traditional interpolation methods, for example, nearest-neighbor, linear, bicubic, or radial basis functions (RBF), typically exhibit limitations such as low accuracy and difficulty in resolving complex flows, especially when the observations x_{obs} are very sparse.

Our research aims to leverage NN architecture to effectively enhance the interpolation operator $I(\cdot)$ in order to improve the performance of AOT nudging on sparse observation. We propose a novel two-stage NN framework to achieve this goal. In the first stage, we utilize a DeepONet [2] trained on fully observed dataset, which enables the network to identify a robust functional basis and provides accurate ground-truth weight coefficients for data reconstruction. The second stage employs the same architecture but with a transformer-based encoder, specifically taken from OFormer model [3], to encode sparse inputs efficiently. The new encoder is coupled with the previously trained decoder from the first stage, now frozen to ensure consistency in coefficient generation. In stage two, the trunk network derived from stage one is also frozen, directing the learning exclusively toward accurate coefficient prediction. Our training objective in the second stage incorporates a loss function composed of two components: the difference between the true coefficients derived from the fully observed model and the predicted coefficients from the partially observed model, and the similarity of reconstructed results to the data measured at sparse locations.

Our method was evaluated on the Quasi-Geostrophic flow [4], which is a simplified version of the Navier-Stokes equations, describing meso-scale ocean dynamics. Our method demonstrated superior performance by effectively reconstructing complete data fields from sparse observations of only 3–5% of total points. Our tests show marked improvements over OFormer and RBF baseline interpolation methods, offering increased accuracy and faster inference.

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